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ON CERTAIN INDICATIONS OF STABILITY WITH TWO LIAPUNOV FUNCTIONS

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We use a Liapunov function with a derivative of constant signs to analyze the problem of asymptotic stability and of instability of an unperturbed motion. We generalize two theorems due to Matrosov [1] for a system of equations of perturbed motion, the right-hand sides of which depend indefinitely on time. The results obtained are also formulated with respect to a part of the variables.

1. Let the following system of equations of perturbed motion be given:

$$\begin{gathered} x^{\cdot} = X(t, x) \qquad (X(t, 0) \equiv 0) \\ x = (x_1, \ldots, x_n) \in R^n, \qquad \|x\| = (x_1^2 + \ldots + x_n^2)^{1/2} \end{gathered} \tag{1.1}$$

where the vector function $X(t, x)$ is defined and continuous on the set

$$\Gamma = \{(t, x) : t \geqq 0, \|x\| < H\} \qquad (0 < H \leqq \infty)$$

while the solutions $x = x(t;\ t_0,\ x_0)$ are defined for $t \geqq t_0$ provided that the initial values $x_0 = x(t_0;\ t_0,\ x_0)$ are sufficiently small in the norm and $t_0 \geqq 0$. Let $x,\ y \in R^n$ and $M \subset R^n$. We introduce the following notation:

$$(x, y) = \sum_{i=1}^n x_i y_i, \quad \rho(x, y) = \|x - y\|, \quad \rho(x, M) = \inf\{\rho(x, y) : y \in M\}$$

Definition 1.1. [1]. Let $M \subset R^n$ and the function $U(t, x)$ be defined and continuous on the set

$$\Gamma' = \{(t, x) : t \geqq 0, \|x\| \leqq H'\} \qquad (0 < H' = \text{const} < H)$$

We shall consider that $U(t, x)$ is definitely nonzero ($U(t, x) \neq 0$) in the set $\{(t, x) : (t, x) \in \Gamma',\ x \in M\}$, if for any $\alpha_1,\ \alpha_2\ (0 < \alpha_1 < \alpha_2 < H')$ positive numbers $\beta_1(\alpha_1, \alpha_2) < \alpha_1$ and $\beta_2(\alpha_1,\ \alpha_2)$ exist such that

$$\begin{gathered} |U(t, x)| \geqq \beta_2 \quad \text{for} \quad (t, x) \in \Gamma', \quad \alpha_1 \leqq \|x\| \leqq \alpha_2 \\ \rho(x, M) \leqq \beta_1 \end{gathered}$$

Definition 1.2. We shall say that the function $\xi(t)$, continuous and nonnegative on $[0, \infty)$, belongs to the class F, if it can be represented in the form $\xi(t) = \varphi(t) + \psi(t)$ where the functions $\varphi(t)$ and $\psi(t)$ are continuous and nonnegative; $\varphi(t)$ is bounded on $(0, \infty)$ and

$$\int_0^\infty \psi(t)\,dt < \infty$$

In what follows we shall denote by $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ the functions continuous on the set Γ together with their partial derivatives in $x_1, \ldots, x_n, t$, and we shall denote by $V^{\cdot}(t, \mathbf{x})$ and $W^{\cdot}(t, \mathbf{x})$ their time derivatives in accordance with the system (1.1).

Theorem 1.1. We assume that

$$\xi(t) = \max\{|(\mathbf{x}, \mathbf{X}(t, \mathbf{x}))| : \|\mathbf{x}\| \leqq H'\} \in F \tag{1.2}$$

and that functions $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ exist with the following properties on the set Γ':

1) the function $V(t, x)$ is positive-definite and admits an infinitesimal upper bound;
2) there exists a function $V_*(\mathbf{x})$ continuous for $\|\mathbf{x}\| \leqq H'$ and such that

$$V^{\cdot}(t, \mathbf{x}) \leqq V_*(\mathbf{x}) \leqq 0$$

3) the function $W(t, \mathbf{x})$ is bounded and $W^{\cdot}(t, \mathbf{x})$ is definitely nonzero $(W^{\cdot}(t, \mathbf{x}) \neq 0)$ in the set

$$E(V_* = 0) = \{(t, \mathbf{x}) : (t, \mathbf{x}) \in \Gamma',\ V_*(\mathbf{x}) = 0\}$$

Then the unperturbed motion $\mathbf{x} = 0$ of the system (1.1) is asymptotically stable uniformly in $\mathbf{x}_0$.

Proof. Under the Conditions (1) and (2) the unperturbed motion $\mathbf{x} = 0$ is stable uniformly in t_0 [2], i.e. for any $\varepsilon > 0$, $\delta(\varepsilon) > 0$ can be found such that from $\|\mathbf{x}_0\| < \delta$ follows $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for any $t \geqq t_0$ and for all $t_0 \geqq 0$. From the properties of the asymptotic and the uniform stabilities, the asymptotic stability uniform in $\mathbf{x}_0$ follows [2], therefore to prove the theorem it is sufficient to verify the property of the asymptotic stability. Since the unperturbed motion is uniformly stable, it suffices to prove that for any numbers η $(0 < \eta < \varepsilon)$, $t_0 \geqq 0$, $\mathbf{x}_0$ $(\|\mathbf{x}_0\| < \delta(\varepsilon))$ we can find $T(\eta, t_0, \mathbf{x}_0)$ such that $\|\mathbf{x}(t_0 + T; t_0, \mathbf{x}_0)\| < \delta(\eta)$. Let us assume the opposite, i.e. that η_* $(0 < \eta_* < \varepsilon)$, $t_* \geqq 0$, $\mathbf{x}_*$ $(\|\mathbf{x}_*\| < \delta(\varepsilon))$ exist such that

$$\mathbf{x}(t) = \mathbf{x}(t;\ t_*,\ \mathbf{x}_*) \in H(\alpha, \varepsilon) = \{\mathbf{x} : \alpha = \delta(\eta_*) \leqq \|\mathbf{x}\| \leqq \varepsilon\}$$

for any $t \geqq t_*$. Then

a) by virtue of Condition (3) $\gamma > 0$ and $\beta > 0$ exist such that

$$|W^{\cdot}(t, \mathbf{x})| \geqq \beta \quad \text{for } t \geqq 0,\ \mathbf{x} \in H(\alpha, \varepsilon) \cap \overline{S(\gamma)}$$
$$S(\gamma) = \{\mathbf{x} : \rho(\mathbf{x}, \{\mathbf{x} : V_*(\mathbf{x}) = 0\}) < \gamma\}$$

hence from $|W(t, \mathbf{x})| < K$ $((t, \mathbf{x}) \in \Gamma')$ it follows that the point $\mathbf{x}(t)$ cannot constantly remain in the set $H(\alpha, \varepsilon) \cap \overline{S(\gamma)}$ during the interval of time equal to $2K / \beta$, and

b) $\nu = \max\{V_*(\mathbf{x}) : \mathbf{x} \in H(\alpha, \varepsilon) \setminus S(\gamma / 2)\} < 0$

consequently the sum of the time intervals during which the point $\mathbf{x}(t)$ remains in the set $H(\alpha, \varepsilon) \setminus S(\gamma / 2)$ is less than $V(t_*, \mathbf{x}_*) / (-\nu)$.

By virtue of (a) and (b) there exists a sequence of numbers $0 \leqq t_1' < t_1'' < \ldots < t_k' < t_k'' < \ldots$ such that

$$\sum_{k=1}^{\infty} (t_k'' - t_k') < \infty \tag{1.3}$$

$$\rho(\mathbf{x}(t_k'), \{\mathbf{x} : V_*(\mathbf{x}) = 0\}) = \gamma/2, \quad \rho(\mathbf{x}(t_k''), \{\mathbf{x} : V_*(\mathbf{x}) = 0\}) = \gamma \tag{1.4}$$

Moreover, making use of (1.3) we find that from (1.2) follows

$$|\,\|\mathbf{x}(t_k'')\|^2 - \|\mathbf{x}(t_k')\|^2\,| \leqq \int_{t_k'}^{t_k''} \left| \frac{d}{dt} \|\mathbf{x}(t)\|^2 \right| dt \leqq \tag{1.5}$$

$$2 \int_{t_k'}^{t_k''} |(\mathbf{x}(t), \mathbf{X}(t, \mathbf{x}(t)))|\, dt \to 0, \quad k \to \infty$$

On the other hand, using (1.4) we obtain

$$|\,\|\mathbf{x}(t_k'')\|^2 - \|\mathbf{x}(t_k')\|^2\,| \geqslant 2\alpha\gamma/2 = \alpha\gamma > 0 \quad (k = 1, 2, \ldots)$$

which contradicts the inequality (1.5), thus proving the theorem.

Note 1.1. The Theorem 1.1 is a generalization of the theorem stated in [1], where the condition (1.2) is replaced by the assumption that the function $\|\mathbf{X}(t, \mathbf{x})\|$ is bounded on the set Γ'.

Example. Suppose we have the following system of equations of perturbed motion:

$$x^{\cdot} = -(1 + \psi(t))\,x, \quad y^{\cdot} = x - y/2, \quad \int_0^{\infty} \psi(t)\,dt < \infty \tag{1.6}$$

where the function $\psi(t)$ is continuous and nonnegative on $[0, \infty]$.

Let us consider the functions $V(x, y) = x^2 + y^2$ and $W(y) = y^2$. By virtue of the system (1.6), their derivatives are

$$V^{\cdot}(t, x, y) = -2(1 + \psi(t))x^2 + 2xy - y^2 \leqq -(x - y)^2 = V_*(x, y)$$
$$W^{\cdot}(x, y) = y\,(2x - y)$$

Moreover, the set $E(V_* = 0) = \{(t, x, y) : t \geqq 0, x = y\}$ and $y(2x - y) \neq 0$ is defined on this set.

Applying Theorem 1.1, we conclude that the solution $x = y = 0$ of the system (1.6) is asymptotically stable uniformly in (x_0, y_0).

Theorem 1.2. Let the condition (1.2) hold and let the functions $V(t, \mathbf{x})$ and $W(t, \mathbf{x})$ exist and have the following properties on the set Γ':

1) the function $V(t, \mathbf{x})$ admits an infinitesimal upper bound and for any value $t_0 \geqq 0$ in an arbitrarily small neighborhood of $\mathbf{x} = 0$ a point $\mathbf{x}_0$ can be found such that $V(t_0, \mathbf{x}_0) > 0$;

2) there exists a function $V_*(\mathbf{x})$ continuous for $\|\mathbf{x}\| \leqq H'$ and such that $V^{\cdot}(t, \mathbf{x}) \geqq V_*(\mathbf{x}) \geqq 0$;

3) the function $W(t, \mathbf{x})$ is bounded and $W^{\cdot}(t, \mathbf{x})$ is definitely nonzero ($W(t, \mathbf{x}) \neq 0$) in the set $E(V_* = 0)$. Then the unperturbed motion $\mathbf{x} = 0$ of system (1.1) is unstable.

Proof. We assume that the solution $\mathbf{x} = 0$ is stable, i.e. for any $\varepsilon > 0$, $t_0 \geqq 0$ such $\delta(\varepsilon, t_0) > 0$ can be found that from $\|\mathbf{x}_0\| < \delta$ follows $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for any $t \geqq t_0$. Let us fix the values of $t_* \geqq 0$ and $\varepsilon > 0$. By virtue of Condition (1) there esists $\mathbf{x}_*$ ($\|\mathbf{x}_*\| < \delta(\varepsilon, t_*)$ such that $V(t_*, \mathbf{x}_*) > 0$.

Let us consider the motion $\mathbf{x}(t) = \mathbf{x}(t;\ t_*,\ \mathbf{x}_*)$. The function $V(t,\ \mathbf{x})$ admits an infinitesimal upper bound, therefore $\alpha > 0$ exists such that $V(t,\ \mathbf{x}) < V(t_*\ \mathbf{x}_*)$ for all $\mathbf{x}\ (\|\mathbf{x}\| < \alpha)$ and $t \geqslant 0$. By virtue of Condition (2) $V(t,\ \mathbf{x}(t)) \geqq V(t_*,\ \mathbf{x}(t_*)) = V(t_*,\ \mathbf{x}_*)$, consequently $\mathbf{x}(t) \in H(\alpha,\ \varepsilon)$ for any $t \geqq t_*$. But this leads to contradiction just as in the proof of Theorem 1.1. This proves Theorem 1.2.

2. The results obtained in Sect. 1 can also be formulated for the stability with respect to a part of the variables assuming them, for the sake of definiteness, to be $x_1, \ldots, x_m$ $(m > 0,\ n = m + p,\ p \geqq 0)$. We introduce the notation [3]

$$y_i = x_i, \quad Y_i(t,\ \mathbf{x}) = X_i(t,\ \mathbf{x}) \quad (i = 1, \ldots, m)$$

$$z_j = x_{m+j}, \quad Z_j(t,\ \mathbf{x}) = X_{m+j}(t,\ \mathbf{x}) \quad (j = 1, \ldots, p)$$

$$\mathbf{y} = (y_1, \ldots, y_m), \quad \mathbf{z} = (z_1, \ldots, z_p)$$

$$\|\mathbf{y}\| = \Big(\sum_{i=1}^{m} y_i^2\Big)^{1/2}, \quad \|\mathbf{z}\| = \Big(\sum_{j=1}^{p} z_i^2\Big)^{1/2}$$

$$E_z(t;\ t_0) = \{\mathbf{z} : \mathbf{z} = \mathbf{z}(t;\ t_0,\ \mathbf{x}_0),\ \|\mathbf{x}_0\| \leqq H'\} \quad (t \geqq t_0 \geqq 0,\ 0 < H' < H)$$

$$\Gamma_y' = \{(t,\mathbf{x}) : t \geqq 0,\ \|\mathbf{y}\| \leqslant H',\ \mathbf{z} \in \bigcup_{t_0 \geqslant 0} \bigcup_{t \geqslant t_0} E_z(t;t_0)\}$$

Theorem 2.1. Assume that for any $t_0 \geqq 0$ the function

$$\xi(t;\ t_0) = \sup\{|(\mathbf{y},\ \mathbf{Y}(t,\ \mathbf{x}))| : \|\mathbf{y}\| \leqq H',\ \mathbf{z} \in E_z(t;\ t_0)\} \in F \qquad (2.1)$$

and also that functions $V(t,\ \mathbf{x})$ and $W(t,\ \mathbf{x})$ exist which have the following properties on the set Γ_y':

1) the function $V(t,\ \mathbf{x})$ is positive-definite and admits an infinitesimal upper bound in $\mathbf{y}$;

2) there exists a function $V_*(\mathbf{y})$, continuous for $\|\mathbf{y}\| \leqq H'$ and such that $V^\cdot(t,\ \mathbf{x}) \leqslant V_*(\mathbf{y}) \leqq 0$;

3) the function $W(t,\ \mathbf{x})$ is bounded and $W^\cdot(t,\ \mathbf{x})$ is definitely nonzero ($W^\cdot(t,\mathbf{x}) \mp 0$) in the set

$$E(V_* = 0) = \{(t,\ \mathbf{x}) : (t,\ \mathbf{x}) \in \Gamma_y',\ V_*(\mathbf{y}) = 0\}$$

Then the unperturbed motion $\mathbf{x} = 0$ of the system (1.1) is asymptotically $\mathbf{y}$-stable uniformly in $\mathbf{x}_0$ [4].

Note 2.1. Theorem 2.1 is a generalization of the Peiffer (*) theorem in which the condition (2.1) is replaced by the assumption that the function $\|\mathbf{Y}(t,\ \mathbf{y},\ \mathbf{z})\|$ is bounded on the set

$$E(t_0) = \{(t,\ \mathbf{x}) : t \geqq t_0,\ \mathbf{x} = \mathbf{x}(t;\ t_0,\ \mathbf{x}_0),\ \|\mathbf{x}_0\| \leqq H'\}$$

for any $t_0 \geqq 0$.

Theorem 2.2. Assume that for any $t_0 \geqq 0$ the condition (2.1) holds and also that functions $V(t,\ \mathbf{x})$ and $W(t,\ \mathbf{x})$ exist with the following properties on the set Γ_y':

1) the function $V(t,\mathbf{x})$ admits an infinitesimal upper bound in $\mathbf{y}$ and for any $t_0 \geqslant 0$ a point $\mathbf{x}_0$ can be found in an arbitrarily small neighborhood of $\mathbf{x} = 0$ such that $V(t_0,\ \mathbf{x}_0) > 0$;

2) there exists a function $V_*(\mathbf{y})$ continuous for $\|\mathbf{y}\| \leqq H'$ and such that $V^\cdot(t,\ \mathbf{x}) \geqq V_*(\mathbf{y}) \geqq 0$;

*) Peiffer, K. La méthode directe de Liapounoff [Liapunov] appliquée à l'étude de la stabilité partielle. (Dissertation). Université Catholique de Louvain, 1968.

3) the function $W(t, x)$ is bounded and $W^{\cdot}(t, x)$ is definitely nonzero $(W^{\cdot}(t,x) \neq 0)$ in the set E $(V_* = 0)$.

Then the unperturbed motion $x = 0$ of the system (1.1) is y-unstable. The proofs of Theorems 2.1 and 2.2 represent modifications of the corresponding proofs given in Sect. 1.

Note 2.2. Analyzing the proofs of these theorems we see that they remain valid if the class of function F (see the definition 1.2) is extended in the following manner: a continuous function $\xi(t)$ nonnegative on $[0, \infty]$ belongs to the class F if for any infinite system of mutually nonintersecting intervals $\{(t_k', t_k'')\}_{k=1}^{\infty}$ with the measure $(t_1'' - t_1') + (t_2'' - t_2') + \ldots < \infty$ the condition

$$\lim_{k\to\infty} \int_{t_k'}^{t_k''} \xi(t)\,dt = 0$$

holds.

3. Applications to mechanical systems. Let the equations of an nonstationary mechanical system acted upon by the potential, gyroscopic and dissipative forces with full dissipation, be given by

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{n} g_{ij} q_j^{\cdot} + \frac{\partial U}{\partial q_i} - \frac{\partial R}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \tag{3.1}$$

$$T = T(\mathbf{q}, \mathbf{q}^{\cdot}) = {}^1/_2\,(A(\mathbf{q})\,\mathbf{q}^{\cdot}, \mathbf{q}^{\cdot}) \geqq {}^1/_2\,\mu\,\|\mathbf{q}^{\cdot}\|^2, \quad \mu = \text{const} > 0$$

$$R = R(t, \mathbf{q}, \mathbf{q}^{\cdot}) = {}^1/_2(B(t, \mathbf{q})\mathbf{q}^{\cdot}, \mathbf{q}^{\cdot}) \geqq {}^1/_2\beta\,\|\mathbf{q}^{\cdot}\|^2, \quad \beta = \text{const} > 0$$

$$\mathbf{q} = (q_1, \ldots, q_n) \in R^n, \quad \mathbf{q}^{\cdot} = (q_1^{\cdot}, \ldots, q_n^{\cdot}) \in R^n$$

where the matrices $A(\mathbf{q}) = [a_{ij}(\mathbf{q})]$ and $B(t, \mathbf{q}) = [b_{ij}(t, \mathbf{q})]$ are symmetric, while the matrix $G(t, \mathbf{q}) = [g_{ij}(t, \mathbf{q})]$ is skew symmetric $(i, j = 1, \ldots, n)$. The gyroscopic coefficients g_{ij} and the dissipation function coefficients b_{ij} are holomorphic functions of $\mathbf{q}$ with the coefficients continuous on $[0, \infty]$. The coefficients a_{ij} and the force function $U(\mathbf{q})$ are assumed holomorphic and

$$U(\mathbf{q}) = \sum_{k=m}^{\infty} U_k(\mathbf{q}) \quad (m \geqslant 2), \quad U(0) = 0$$

where $U_k(\mathbf{q})$ is a homogeneous function of kth degree.

Theorem 3.1. We denote by $\lambda(t, \mathbf{q})$ the largest absolute eigenvalue of the symmetric matrix ${}^1/_2[(G - B)A^{-1} - A^{-1}(G + B)]$. We assume that the function

$$\xi(t) = \max\{|\lambda(t, \mathbf{q})| : \|\mathbf{q}\| \leqq H'\} \in F, \qquad 0 < H = \text{const} \tag{3.2}$$

and that a function $\zeta(t) \in C^1[0, \infty)$ exists such that

$$\zeta(t) \geqq \max\left\{\sum_{i,j=1}^{n} |g_{ij}(t, \mathbf{q}) - b_{ij}(t, \mathbf{q})| : \|\mathbf{q}\| \leqq H'\right\} \tag{3.3}$$

$$\left|\frac{d}{dt}\zeta(t)\right| [\zeta(t)]^{-2} < K \quad (t \geqq 0, \ K = \text{const})$$

$$\int_0^{\infty} \frac{1}{\zeta(t)}\,dt = \infty$$

Then

a) if the force function $U(\mathbf{q})$ is negative-definite and the sum

$$S = \sum_{k=m}^{\infty} kU_k(\mathbf{q})$$

is sign-definite, then the unperturbed motion $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ of the system (3.1) is asymptotically stable uniformly in $\{\mathbf{q}_0, \mathbf{q}_0^{\cdot}\}$;

b) if the function $U(\mathbf{q})$ can assume a positive value in an arbitrarily small neighborhood of the point $\mathbf{q} = 0$ and the sum S is sign-definite, then the unperturbed motion $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ of the system (3.1) is unstable.

Proof. Passing from the Lagrangian to the Hamiltonian variables, we obtain the equation of motion (3.1) in the form

$$q_i^{\cdot} = \frac{\partial H}{\partial p_i}, \quad p_i^{\cdot} = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} g_{ij}\frac{\partial H}{\partial p_j} - \sum_{j=1}^{n} b_{ij}\frac{\partial H}{\partial p_j} \tag{3.4}$$

$$\mathbf{p} = A(\mathbf{q})\mathbf{q}^{\cdot}, \quad H(\mathbf{p}, \mathbf{q}) = T - U = (\mathbf{p}, A^{-1}(\mathbf{q})\mathbf{p}) - U(\mathbf{q})$$

Condition (3.2) implies that condition (1.2) holds for the system (3.4).

Let us now consider the functions

$$V(\mathbf{q}, \mathbf{q}^{\cdot}) = T - U, \; W(t, \mathbf{q}, \mathbf{q}^{\cdot}) = \frac{1}{\zeta(t)}\sum_{i=1}^{n}\frac{\partial T}{\partial q_i^{\cdot}} q_i$$

By virtue of the system (3.4), their derivatives are

$$V^{\cdot}(t, \mathbf{q}, \mathbf{q}^{\cdot}) = -2R \leqq -\beta\,\|\mathbf{q}^{\cdot}\|^2$$

$$W^{\cdot}(t, \mathbf{q}, \mathbf{q}^{\cdot}) = \frac{1}{\zeta(t)}\Big[2T + (G\mathbf{q}, \mathbf{q}^{\cdot}) + (\operatorname{grad} U, \mathbf{q}) + \sum_{i=1}^{n} q_i\Big(\frac{\partial T}{\partial q_i} - \frac{\partial R}{\partial q_i^{\cdot}}\Big)\Big] - \frac{\zeta^{\cdot}(t)}{[\zeta(t)]^2}\sum_{i=1}^{n}\frac{\partial T}{\partial q_i^{\cdot}} q_i$$

On the set $E(\beta\,\|\mathbf{q}^{\cdot}\|^2 = 0)$ we have

$$\zeta(t)\, W^{\cdot}(t, \mathbf{q}, \mathbf{q}^{\cdot}) = \sum_{k=m}^{\infty} kU_k(\mathbf{q})$$

$$E(\beta\,\|\mathbf{q}^{\cdot}\|^2 = 0) = \{(t, \mathbf{q}, \mathbf{q}^{\cdot}) : t \geqq 0, \|\mathbf{q}\| \leqq H', \mathbf{q}^{\cdot} = 0\}$$

Using the property of sign-definiteness of this function and the condition (3.3) we conclude, that for any α_1, α_2 $(0 < \alpha_1 < \alpha_2 < H')$, $\beta(\alpha_1, \alpha_2) > 0$ exists such that

$$\int_0^{\infty} \min\{|W^{\cdot}(t, \mathbf{q}, \mathbf{q}^{\cdot})| : \alpha_1^2 \leqq \|\mathbf{q}\|^2 + \|\mathbf{q}^{\cdot}\|^2 \leqq \alpha_2, \|\mathbf{q}^{\cdot}\| \leqq \beta\}\, dt = \infty$$

From the point (a) of the proof of Theorem 1.1 we see that in Theorems 1.1 and 1.2, this property can be substituted for the definiteness of $W^{\cdot} \neq 0$ in the set $E(\beta\,\|\mathbf{q}^{\cdot}\|^2 = 0)$ (see [1]), therefore Theorem 3.1. follows from the results of Sect. 1.

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MOTION OF A HIGHLY VISCOUS LIQUID IN A ROTATING TORUS

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We consider a motion of a viscous incompressible liquid in a toroidal cavity within a top spinning with an arbitrary angular velocity and acceleration. The results obtained can be used to determine the position of the toroidal tube filled with a viscous liquid relative to the top axes of inertia, which will minimize the time necessary to stabilize the motion of the top.

1. Statement of the problem. It was shown in [1] for the motion of a solid with cavities completely filled with a viscous liquid, that for the first approximation to the value of the Reynolds number $R = l^2 / T\nu \ll 1$ and for large values of time $t > l^2/\nu$, the contribution of the relative motion of the liquid to the moment of impulse of the solid-liquid system does not depend on the initial motion of the liquid and can be written in the form

$$\mathbf{L} = -\frac{\rho}{\nu} \sum_{i,j=1}^{3} P_{ij}\varepsilon_i(t)\,\mathbf{e}^{(j)}, \qquad P_{ij} = -\int_V \mathbf{e}^{(j)}[\mathbf{r}, \boldsymbol{\zeta}^{(i)}]\,dV \tag{1.1}$$

where the integration is performed over the volume of the cavity, ε is the angular acceleration of the solid and $\zeta^{(i)}$ is the solution of the system (see [1])

$$\Delta\boldsymbol{\zeta}^{(i)} = \nabla s^{(i)} + [\mathbf{e}^{(i)}, \mathbf{r}], \qquad \operatorname{div}\boldsymbol{\zeta}^{(i)} = 0, \qquad \boldsymbol{\zeta}^{(i)}|_S = 0 \tag{1.2}$$

When time is large, the quantities $\boldsymbol{\zeta}^{(i)}$ and $s^{(i)}$ determine the velocity $\mathbf{u}$ of the liquid relative to the solid, and its generalized pressure p

$$\mathbf{u} = \frac{1}{\nu}\sum_{i=1}^{3} \varepsilon_i\boldsymbol{\zeta}^{(i)}, \qquad p = \sum_{i=1}^{3} \varepsilon_i s^{(i)}$$

In [1] we find the values of P_{ij} computed for a sphere, an ellipsoid and a cylinder. Below we consider the case of a toroidal cavity, representing the simplest example of a doubly connected region.

2. Investigation of equations of motion of a liquid in a torus. Let the cavity have the form of a torus with the median line radius denoted by R and the tube radius by r. We introduce the intrinsic coordinate system of the torus with its